Section 1-3, Mathematics 104

Integer Exponents

We define an integer exponent n over a base $a \neq 0$ as follows:

$$a^n = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}}$$

So if we multiply two exponential expressions with the same base

$$a^n a^m = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}} \times \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{m \text{ times}} = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n+m \text{ times}} = a^{n+m}$$

(1) $\underline{a^n a^m = a^{n+m}}$

Similarly if $n > m$ we can divide

$$\frac{a^n}{a^m} = \frac{a \cdot a \cdot a \cdots a \cdot a \; n-times}{a \cdot a \cdot a \cdots a \cdot a \; m-times} = \left\{ \frac{a}{a} \cdot \frac{a}{a} \cdot \frac{a}{a} \cdots \frac{a}{a} (m-times) \right\} \cdot \frac{a \cdot a \cdot a \cdots a}{(n-m) times} = a^{n-m}$$

(2) $\underline{\frac{a^n}{a^m} = a^{n-m}}$

Note that $\frac{a^n}{a^n} = 1$ so applying the previous equation we see that $1 = \frac{a^n}{a^n} = a^{n-n} = a^0$

(3) for $\underline{a \neq 0 \quad a^0 = 1}$

Using this preceding

$$\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$$

(4) $\frac{1}{a^n} = a^{-n}$

Other properties of exponents

$$\left(a^n\right)^m = \underbrace{\left(\begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ n-times \end{matrix}\right)\cdot\left(\begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ n-times \end{matrix}\right)\cdots\left(\begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ n-times \end{matrix}\right)}_{m-times} = \begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ nm-times \end{matrix} = a^{nm}$$

(5) $\underline{\left(a^n\right)^m = a^{nm}}$

$$\left(ab\right)^n = \begin{matrix} ab\cdot ab\cdots ab \\ n-times \end{matrix} = \begin{matrix} a\cdot a\cdots a \\ n-times \end{matrix} \times \begin{matrix} b\cdot b\cdots b \\ n-times \end{matrix} = a^n b^n$$

(6) $\underline{\left(ab\right)^n = a^n b^n}$

$$\left(\frac{a}{b}\right)^n = \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots\frac{a}{b} = \frac{a\cdot a\cdot a\ldots a}{b\cdot b\cdot b\ldots b} = \frac{a^n}{b^n}$$

(7) $\underline{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$

$$\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{\frac{1}{a^n}}{\frac{1}{b^n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$$

(8) $\underline{\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n}$

$$\frac{a^{-n}}{b^{-m}} = \frac{\frac{1}{a^n}}{\frac{1}{b^m}} = \frac{b^m}{a^n}$$

(9) $\underline{\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}}$

Summary

$a^n a^m = a^{n+m}$	$\frac{a^n}{a^m} = a^{n-m}$	for $a \neq 0 \quad a^0 = 1$
$\frac{1}{a^n} = a^{-n}$	$\left(a^n\right)^m = a^{nm}$	$\left(ab\right)^n = a^n b^n$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$

Example

Simplify $\left(\frac{6}{5}\right)^3 \left(\frac{3^3 5}{2}\right)^4$

$$\left(\frac{6}{5}\right)^3 \left(\frac{3^3 5}{2}\right)^4 = \frac{6^3}{5^3} \cdot \frac{\left(3^3\right)^4 5^4}{2^4} = \frac{6^3 3^{12} 5^4}{5^3 2^4} = \frac{2^3 3^3 3^{12} 5^4}{5^3 2^4} = \frac{3^{15} 5}{2}$$

Section 1-4, Mathematics 104

Language

When I was growing up I was sent to Hebrew school where I was subjected to a very strange experience. I was taught the Hebrew Alphabet and I was taught to read Hebrew, but I mean this literally. I knew how to read Hebrew words out loud but had no idea what they meant.

The purpose of this strange exercise was so that I could read prayer sin Hebrew, although for the most part I had no idea what I was saying.

I'm not telling you this because think this was a good or bad idea. Just so you will think about the fact that there is more than one way to know a language.

In the case of the language of mathematics, it is possible to know how to manipulate the language without knowing what it means. The opposite is also possible.

When you are learning a new language, it is best to know as much about the language as possible, for example the meaning of words, spelling, syntax, grammar, and sentence structure. This is what I want to start working today.

The first few classes we talked about numbers, eg.

$$17, \quad \frac{3}{5} \quad 6.25 \quad 3.3\bar{3} \quad \sqrt{2} \quad \pi$$

What do each of these mean?

The next part of the language is what is called an operator.

$$+ \quad - \quad \times \quad \div \quad \sqrt{}$$

Note that there are some confusing issues.

The - in -3 is not the same as 12-3. The former is a unary operator, meaning it operates on a single number. The square root sign is also unary. Exponentiation is likewise unary. The other operators are binary in that they operate on two numbers.